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The first eigenvalue of the Dirac operator on locally reducible Riemannian manifolds

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Abstract

We prove a lower estimate for the first eigenvalue of the Dirac operator on a compact locally reducible Riemannian spin manifold with positive scalar curvature. We determine also the universal covers of the manifolds on which the smallest possible eigenvalue is attained.

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1. Introduction

Let M be a compact Riemannian spin manifold with positive scalar curvature s. A well known consequence of the Schrödinger–Lichnerowicz formula [12,10]

$$D^2 = \nabla^* \nabla + \frac{s}{4} \tag{1.1}$$

is the Lichnerowicz vanishing theorem [10]: the kernel of the Dirac operator D coincides with the space of parallel spinors. If this kernel is non-trivial, then the scalar curvature must be zero and, even more, M must be Ricci-flat. The simply connected irreducible manifolds which admit parallel spinors were described by Wang [13]. They are exactly the manifolds whose holonomy group is one of $SU(n)$, $Sp(n)$, G_2, $Spin(7)$. For each of these cases Wang found the dimension of the space of parallel spinors.

If the scalar curvature is not identically zero, then the kernel of D is trivial. This holds true, in particular, if $s > 0$, which we shall assume from now on. In this case an estimate for the square of the first eigenvalue λ of the Dirac operator was obtained by Friedrich [4]:

$$\lambda^2 \geq \frac{n}{n-1} \cdot \frac{\min s}{4}, \tag{1.2}$$

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where n is the dimension of the manifold. The limiting case of this estimate (i.e., the case of equality in (1.2)) is characterized by the existence of a non-trivial real Killing spinor on M. It was shown by Bär [3] that the last property is equivalent to the existence of a parallel spinor on the cone over M. Thus, using Wang's results, he classified the simply connected manifolds admitting real Killing spinors. They are the sphere S^n, the Einstein–Sasakian manifolds, the 3-Sasakian manifolds, the 6-dimensional nearly Kähler manifolds and the 7-dimensional manifolds with nearly parallel vector cross product [3].

The manifolds with non-trivial real Killing spinors have constant positive scalar curvature. They are furthermore Einstein, locally irreducible and their holonomy is $SO(n)$. In particular, they do not admit any parallel k-form for $k \neq 0, n$ [5]. This shows that the estimate (1.2) cannot be sharp, for example, on Kähler or quaternionic Kähler manifolds. Indeed, better estimates have been proved in these cases by Kirchberg [7,8] and Kramer et al. [9] respectively.

Another situation in which (1.2) is not sharp was considered by the author, G. Grantcharov and S. Ivanov. In [1] they showed that if M admits a non-trivial parallel 1-form, then

$$\lambda^2 \geq \frac{n-1}{n-2} \cdot \frac{\min s}{4} \tag{1.3}$$

and the universal cover of a manifold, on which the first eigenvalue is the smallest possible, is a Riemannian product $\mathbb{R} \times N$, where N is a simply connected manifold with a real Killing spinor. (In fact, it was proved by Moroianu and Ornea [11] that the same holds true under the weaker assumption that the 1-form is harmonic and has constant length.)

A manifold with a non-trivial parallel 1-form is locally a product of a 1-dimensional and an $(n-1)$-dimensional manifold. In [6] Kim considered the more general situation where the manifold is locally a product of Riemannian manifolds of arbitrary dimensions n_1 and n_2. More precisely, he proved the following generalization of (1.3). Let the tangent bundle $TM = T_1 \oplus T_2$, where T_1 and T_2 are parallel, $\dim T_i = n_i$, $i = 1, 2$, and $n_1 \leq n_2$. Then the first eigenvalue of the Dirac operator satisfies

$$\lambda^2 \geq \frac{n_2}{n_2-1} \cdot \frac{\min s}{4}. \tag{1.4}$$

Kim also gave examples of manifolds on which the equality in (1.4) is attained. These are global products $M = M_1 \times M_2$, where $\dim M_1 = n_1$, $\dim M_2 = n_2$, M_2 admits a real Killing spinor and M_1 has a parallel spinor if $n_1 < n_2$ or either a parallel or a real Killing spinor if $n_1 = n_2$. In [6] Kim stated the question of whether these examples give the general form of the universal coverings of the manifolds on which the limiting case of (1.4) occurs.

In the present paper we show that the answer to this question is positive. More generally, we consider manifolds which are locally a product of k Riemannian manifolds and find an estimate for the first eigenvalue of the Dirac operator. We also describe the universal covers of the manifolds on which the limiting case of this estimate is attained. We summarize our results in the following theorem.

Theorem 1.1. *Let M be a compact Riemannian spin manifold with positive scalar curvature s. Let $TM = T_1 \oplus \cdots \oplus T_k$, where T_i are parallel distributions of dimension n_i, $i = 1, \ldots, k$, and $n_1 \leq \cdots \leq n_k$. Then the first eigenvalue λ of the Dirac operator satisfies*

$$\lambda^2 \geq \frac{n_k}{n_k-1} \cdot \frac{\min s}{4}. \tag{1.5}$$

If the equality in (1.5) is attained, then the universal cover $\widetilde{M}$ of M is isometric to a product $M_1 \times \cdots \times M_k$, where $\dim M_i = n_i$, M_k has a real Killing spinor and for $i < k$ M_i has a parallel spinor if $n_i < n_k$ or either a parallel spinor or a real Killing spinor if $n_i = n_k$.

Notice that the estimate (1.5) contains (1.2)–(1.4) as special cases. We remark also that $n_k > 1$ because of the positivity of the scalar curvature and that compact manifolds on which (1.5) is an equality do exist. For example, $M = M_1 \times \cdots \times M_k$, where M_i, $i = 1, \ldots, p$, is a compact manifold admitting a parallel spinor (e.g., $M_i = S^1 \times \cdots \times S^1$ (n_i times)) and M_j, $j = p+1, \ldots, k$, is a compact manifold of dimension n_k admitting a real Killing spinor (e.g., $M_j = S^{n_k}$).

The proof of (1.5) uses a typical technique for such situations. We introduce certain suitable twistor operator and prove a Weitzenböck formula including this operator and the Dirac operator, which in its turn implies (1.5). This simple proof is somewhat different than the proof of (1.4) in [6] but its advantage is that it allows us to handle easily the limiting case.

2. Preliminaries

Let M be a compact Riemannian spin manifold such that the tangent bundle

$$TM = T_1 \oplus \cdots \oplus T_k, \tag{2.6}$$

where T_i are parallel and pairwise orthogonal distributions of dimension n_i, $i = 1, \dots, k$.

We denote the pointwise inner products by $\langle\cdot,\cdot\rangle$ and the corresponding norms by $|\cdot|$. The global inner products will be denoted by $(\cdot,\cdot)$, $(\cdot,\cdot) = \int_M \langle\cdot,\cdot\rangle \mathrm{d}Vol_M$, and the corresponding global norms by $\|\cdot\|$.

By $e_{1,1}, \dots, e_{1,n_1}, \dots, e_{k,1}, \dots, e_{k,n_k}$ we will denote an adapted local orthonormal frame, i.e., such that $e_{i,1}, \dots, e_{i,n_i}$ spans T_i. The dual frame will be denoted by $e^{1,1}, \dots, e^{1,n_1}, \dots, e^{k,1}, \dots, e^{k,n_k}$. Since $T_1, \dots, T_k$ are parallel, we can always assume that $\nabla e_{i,l} = 0$ at a fixed point x.

Another consequence of the parallelism of T_i is that $R(X, Y) = 0$ whenever $X \in T_i$, $Y \in T_j$ with $i \neq j$.

Let s_i be the "scalar curvature" of T_i, i.e.,

$$s_i = \sum_{l,m=1}^{n_i} \langle R(e_{i,l}, e_{i,m})e_{i,m}, e_{i,l}\rangle.$$

Hence the scalar curvature s of M is $s = \sum_{i=1}^k s_i$.

We denote by p_i the orthogonal projections from TM onto T_i and from T^*M onto T_i^*.

Let ΣM be the spinor bundle of M, $\nabla : \Gamma(\Sigma M) \longrightarrow \Gamma(T^*M \otimes \Sigma M)$ the covariant derivative of the Levi-Civita connection and $\mu : T^*M \otimes \Sigma M \longrightarrow \Sigma M$ the Clifford multiplication (we write also $\alpha \cdot \psi$ instead of $\mu(\alpha \otimes \psi)$). Thus the Dirac operator $D : \Gamma(\Sigma M) \longrightarrow \Gamma(\Sigma M)$ is given by $D = \mu \circ \nabla$, i.e., $D\psi = \sum_{i=1}^k \sum_{l=1}^{n_i} e^{i,l} \cdot \nabla_{e_{i,l}} \psi$.

We define $\nabla_i : \Gamma(\Sigma M) \longrightarrow \Gamma(T_i^* \otimes \Sigma M)$ by $\nabla_{i\,X}\psi := \nabla_{p_i(X)}\psi$ and $D_i : \Gamma(\Sigma M) \longrightarrow \Gamma(\Sigma M)$ by $D_i := \mu \circ \nabla_i$, i.e., $D_i\psi = \sum_{l=1}^{n_i} e^{i,l} \cdot \nabla_{i\,e_{i,l}}\psi$. So we have

$$\nabla = \sum_{i=1}^k \nabla_i, \qquad D = \sum_{i=1}^k D_i.$$

Next we list several formulae whose proofs are straightforward and quite similar to those of the corresponding results about the Dirac operator D.

Let $X \in \Gamma(TM)$ (resp. $\alpha \in \Gamma(T^*M)$) be orthogonal to T_i (resp. T_i^*) and $(\nabla X)_x = 0$ (resp. $(\nabla\alpha)_x = 0$) at a given point x. Then

$$(\nabla_X(D_i\psi))_x = (D_i(\nabla_X\psi))_x, \qquad (\alpha \cdot (D_i\psi))_x = -(D_i(\alpha \cdot \psi))_x. \tag{2.7}$$

This implies, in particular,

$$D_iD_j + D_jD_i = 0 \quad \text{for } i \neq j \tag{2.8}$$

and therefore

$$D^2 = \sum_{i=1}^k D_i^2. \tag{2.9}$$

We also have

$$\|D_i\psi\|^2 = (D_i^2\psi, \psi) \tag{2.10}$$

and the following "partial" Schrödinger–Lichnerowicz formula

$$D_i^2 = \nabla_i^*\nabla_i + \frac{s_i}{4}. \tag{2.11}$$

3. Proof of Theorem 1.1

Let $\pi : T^*M \otimes \Sigma M \longrightarrow T^*M \otimes \Sigma M$ be defined by

$$\pi(\alpha \otimes \psi) = \alpha \otimes \psi + \sum_{i=1}^{k} \frac{1}{n_i} \sum_{l=1}^{n_i} e^{i,l} \otimes (e^{i,l} \cdot p_i(\alpha) \cdot \psi).$$

Clearly, this definition does not depend on the choice of the adapted frame $\{e_{i,l}\}$ and $\pi(T^*M \otimes \Sigma M) \subset \ker \mu$.

We introduce the following "adapted" twistor operator

$$P : \Gamma(\Sigma M) \longrightarrow \Gamma(T^*M \otimes \Sigma M) : \qquad P := \pi \circ \nabla.$$

In other words,

$$P\psi = \nabla\psi + \sum_{i=1}^{k} \frac{1}{n_i} \sum_{l=1}^{n_i} e^{i,l} \otimes (e^{i,l} \cdot D_i\psi).$$

We easily see

$$|P\psi|^2 = |\nabla\psi|^2 - \sum_{i=1}^{k} \frac{1}{n_i} |D_i\psi|^2.$$

Using the Schrödinger–Lichnerowicz formula (1.1) and (2.10), this implies

$$\|P\psi\|^2 = \left(\left(D^2 - \sum_{i=1}^{k} \frac{1}{n_i} D_i^2\right)\psi, \psi\right) - \left(\frac{s}{4}\psi, \psi\right).$$

By (2.9) $D_k^2 = D^2 - \sum_{i=1}^{k-1} D_i^2$. Hence

$$\|P\psi\|^2 = \left(1 - \frac{1}{n_k}\right)(D^2\psi, \psi) - \sum_{i=1}^{k-1} \left(\frac{1}{n_i} - \frac{1}{n_k}\right)(D_i^2\psi, \psi) - \left(\frac{s}{4}\psi, \psi\right)$$

and therefore

$$\|D\psi\|^2 = \frac{n_k}{n_k - 1}\|P\psi\|^2 + \sum_{i=1}^{k-1} \frac{n_k - n_i}{n_k - 1}\|D_i\psi\|^2 + \frac{n_k}{n_k - 1}\left(\frac{s}{4}\psi, \psi\right).$$

Since $n_k \geq n_i$ for each i, we obtain that the first eigenvalue λ of D satisfies (1.5). This proves the first part of the theorem.

Suppose now that for the first eigenvalue λ we have $\lambda^2 = \frac{n_k}{n_k-1} \cdot \frac{\min s}{4}$. Then for each eigenspinor ψ of D for λ we have $P\psi = 0$, $D_i\psi = 0$ if $n_i < n_k$ and s is constant on the support of ψ. But D^2 is an elliptic operator of second order and it follows from [2] that each eigenspinor of D does not vanish on a dense open subset of M. Thus the scalar curvature s is constant on the whole manifold and the parallelism of T_i implies that the same is true for s_i, $i = 1, \dots, k$.

The equation $P\psi = 0$ is equivalent to

$$\nabla_i\psi + \frac{1}{n_i} \sum_{l=1}^{n_i} e^{i,l} \otimes (e^{i,l} \cdot D_i\psi) = 0, \quad i = 1, \dots, k. \tag{3.12}$$

If $D_i\psi = 0$ (in particular, if $n_i < n_k$), then

$$\nabla_i\psi = 0. \tag{3.13}$$

Let $D_i\psi \neq 0$. This means that $n_i = n_k$. We can write (3.12) as

$$\nabla_{iX}\psi + \frac{1}{n_i} p_i(X^\flat) \cdot D_i\psi = 0, \quad X \in TM, \tag{3.14}$$

where $X^\flat$ is the 1-form corresponding to X via the metric. Since $D\psi$ is also an eigenspinor of D for the eigenvalue λ, (3.14) is satisfied by it too:

$$\nabla_{i\,X}(D\psi) + \frac{1}{n_i} p_i(X^\flat) \cdot D_i D\psi = 0, \quad X \in TM.$$

Now, if $X \in \Gamma(TM)$ and $(\nabla X)_x = 0$, then (2.7) and (2.8) imply that at x

$$\nabla_{i\,X}(D_i\psi) + \frac{1}{n_i} p_i(X^\flat) \cdot D_i^2\psi + \sum_{j\neq i} D_j \left(\nabla_{i\,X}\psi + \frac{1}{n_i} p_i(X^\flat) \cdot D_i\psi\right) = 0.$$

This, together with (3.14), shows that

$$\nabla_{i\,X}(D_i\psi) + \frac{1}{n_i} p_i(X^\flat) \cdot D_i^2\psi = 0, \quad X \in TM. \tag{3.15}$$

By a straightforward computation, similar to the case of the usual twistor spinors, (3.14) and (2.11) yield

$$D_i^2\psi = \frac{n_i}{n_i - 1} \cdot \frac{s_i}{4}\psi. \tag{3.16}$$

In particular, by (2.10), $\lambda_i^2 := \frac{n_i}{n_i-1} \cdot \frac{s_i}{4} > 0$.

Let $\varphi_{i\pm} := \psi \pm \frac{1}{\lambda_i} D_i\psi$. By (3.14)–(3.16) we obtain

$$\nabla_{i\,X}\varphi_{i\pm} = \mp\frac{\lambda_i}{n_i} p_i(X^\flat) \cdot \varphi_{i\pm}, \quad X \in TM. \tag{3.17}$$

At least one of φ_{i+} and φ_{i-} is not identically zero because the same is true for ψ. Let $x \in M$ be such that, for example, $\varphi_{i+}(x) \neq 0$. The universal cover $\widetilde{M}$ of M is a Riemannian product $\widetilde{M} = M_1 \times \cdots \times M_k$ because of the decomposition (2.6). Let the point $\widetilde{x} = (x_1, \ldots, x_k)$ project on x. Denote by $f_i : M_i \longrightarrow \widetilde{M}$ the inclusion $f_i(y) = (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k)$. Consider the pull-back $(q \circ f_i)^*\Sigma M$ of ΣM to M_i, where $q : \widetilde{M} \longrightarrow M$ is the projection. The bundle $(q \circ f_i)^*\Sigma M$ is a Clifford module on M_i and is therefore a sum of finitely many, say r_i, copies of the spinor bundle ΣM_i of M_i. Also the Levi-Civita connection on ΣM pulls back to the Levi-Civita connection on $(q \circ f_i)^*\Sigma M = \oplus_{l=1}^{r_i} \Sigma M_i$. Thus (3.17) implies that $(q \circ f_i)^*\varphi_{i+} = (\varphi_1, \ldots, \varphi_{r_i}) \in \Gamma((q \circ f_i)^*\Sigma M)$ and each of $\varphi_1, \ldots, \varphi_{r_i} \in \Gamma(\Sigma M_i)$ is a real Killing spinor on M_i. Since $\varphi_{i+}(x) \neq 0$, at least one of $\varphi_1, \ldots, \varphi_{r_i}$ is not identically zero. Therefore M_i admits a non-trivial real Killing spinor if $D_i\psi \neq 0$.

In a similar way (3.13) implies that M_i admits a non-trivial parallel spinor if $D_i\psi = 0$.

This completes the proof of Theorem 1.1.

References

[1] B. Alexandrov, G. Grantcharov, S. Ivanov, An estimate for the first eigenvalue of the Dirac operator on compact Riemannian spin manifold admitting parallel one-form, J. Geom. Phys. 28 (1998) 263–270.

[2] N. Aronszajn, A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order, J. Math. Pures Appl. 35 (1957) 235–249.

[3] C. Bär, Real Killing spinors and holonomy, Comm. Math. Phys. 154 (1993) 509–521.

[4] T. Friedrich, Der erste Eigenwert des Dirac–Operators einer kompakten, Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung, Math. Nachr. 97 (1980) 117–146.

[5] O. Hijazi, Opérateurs de Dirac sur le variétés riemanniennes. Minoration des valeurs propres, Thèse de 3-eme Cycle, Ecole Polytechnique, 1984.

[6] E.C. Kim, Lower bounds of the Dirac eigenvalues on Riemannian product manifolds, arXiv:math.DG/0402427.

[7] K.-D. Kirchberg, An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds of positive scalar curvature, Ann. Global Anal. Geom. 4 (1986) 291–325.

[8] K.-D. Kirchberg, The first eigenvalue of the Dirac operator on Kähler manifolds, J. Geom. Phys. 7 (1990) 447–468.

[9] W. Kramer, U. Semmelmann, G. Weingart, Eigenvalue estimates for the Dirac operator on quaternionic Kähler manifolds, Math. Z. 230 (4) (1999) 727–751.

[10] A. Lichnerowicz, Spineurs harmoniques, C. R. Acad. Sci. Paris Ser. A 257 (1963) 7–9.

[11] A. Moroianu, L. Ornea, Eigenvalue estimates for the Dirac operator and harmonic 1-forms of constant length, C. R. Math. Acad. Sci. Paris 338 (7) (2004) 561–564.

[12] E. Schrödinger, Diracsches Elektron im Schwerfeld I, Sitzungsber. Preuß. Akad. Wiss., Phys.-Math. Kl. 1932 (11–12) (1932) 105–128.
[13] M. Wang, Parallel spinors and parallel forms, Ann. Global Anal. Geom. 7 (1989) 59–68.